

Contra and Almost D-Precontinuous Functions in D-Metric Spaces

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ABSTRACT

The purpose of this paper is to introduce and investigate weak form of D-precontinuous function in D-metric spaces, namely contra and almost D-precontinuous functions via D-preopen sets. The relationships among this form with the other known functions are introduced.

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Keywords

Open set; Metric spaces, continuous functions.

1. INTRODUCTION

In 1984, Dhage, [3], introduced a new notion of a new structure of D-metric space which is a natural generalization of the notion of ordinary metric space to higher dimensional metric spaces, [4]. In 2000, Dhage, [2], introduced some results in D-metric spaces and the notion of open and closed balls. In 2017, Ali Fora, Massadeh and Bataineh, [1], introduced and a new topological of D-closed set, D-continuous and D-xed point property discussed of its properties, some result for this subject are also established. structure of D-closed set. In 2021, Hussain and Saif, [5], introduce and investigate weak form of D-open sets in D-metric spaces, namely D-preopen sets. The relationships among this form with the other known sets are introduced. They give introduce the notions of the interior operator, the closure operator and frontier operator via D-preopen sets. In 2021, Hussain and Saif, [6], introduce and investigate weak form of D-continuous functions in D-metric spaces, namely D-precontinuous functions .

This paper is organized as follows: Section 2 is devoted to some preliminaries. In Section 3 we introduce the concepts of contra and almost D-precontinuous functions via D-preopen sets. Furthermore, the relationship with the other known sets will be studied.

2. PRELIMINARIES

DEFINITION 2.1. [1]. Let $f : (X, D) \rightarrow (Y, P)$ be a function between two D-metric spaces (X, D) and (Y, P) . Then f is said to be D_P -continuous at $p \in X$ provided that for any sequence $\{x_n\}$ converging in (X, D) to p , then $\{f(x_n)\}$ must converge in (Y, P) to $f(p)$. A function $f : (X, D) \rightarrow (Y, P)$ is called D_P -continuous if f is D_P -continuous at each p in X .

By $O_\varepsilon^D(x)$, we mean the D-open ball with center x and radius $\varepsilon > 0$, that is,

$$O_\varepsilon^D(x) = \{y \in X : d(x, y, y) < \varepsilon\}.$$

By $C_\varepsilon^D(x)$, we mean the D-closed ball with center x and radius $\varepsilon > 0$, that is,

$$C_\varepsilon^D(x) = \{y \in X : d(x, y, y) \leq \varepsilon\}.$$

The set $G \subseteq X$ is called a D-open set in D-metric space (X, D) if for every $x \in G$, there is $\varepsilon > 0$ such that $O_\varepsilon^D(x) \subseteq G$. The set G is called D-closed set in D-metric space (X, D) if $X - G$ is an D-open set in D-metric space (X, D) . For D-metric space (X, D) and $G \subseteq X$, the interior set of G is denoted by $Int_D(G)$ and the clouser set of G is denoted by $Cl_D(G)$.

For D-metric space (X, D) and $G \subseteq X$ is called a D-preopen set, [5], in D-metric space (X, D) if for every $x \in G$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$. The set $G \subseteq X$ is called a D-preclosed set in D-metric space (X, D) if $X - G$ is a D-preopen set in D-metric space (X, D) . Recall, [5], that they introduced the interior operator, the closure operator and frontier operator via D-preopen sets. The set $G \subseteq X$, the D_P -interior set of G is denoted by $Int_P^D(G)$ and the D_P -clouser set of G is denoted by $Cl_P^D(G)$. For a subset G of D-metric space (X, D) , the D-frontier operator of G is defined by

$$\Gamma_P^D(G) = Cl_P^D(G) - Int_P^D(G).$$

DEFINITION 2.2. [1]. Let $f : (X, D) \rightarrow (Y, P)$ be a function between two D-metric spaces (X, D) and (Y, P) . Then f is said to be D_P -weakly continuous at $p \in X$ provided that for any P -open set H in Y containing $f(p)$, there exists a D -open set U in X containing p such that $f(U) \subseteq H$. A function $f : (X, D) \rightarrow (Y, P)$ is called D_P -weakly continuous if f is D_P -weakly continuous at each p in X .

THEOREM 2.3. [1]. Let $f : (X, D) \rightarrow (Y, P)$ be a function between two D-metric spaces (X, D) and (Y, P) . If f is D_P -continuous function then f is D_P -weakly continuous.

THEOREM 2.4. [1]. The followings are equivalent for the function $f : (X, D) \rightarrow (Y, P)$ between two D-metric spaces (X, D) and (Y, P) .

- (1) f is D_P -weakly continuous.
- (2) For any P -open set H in Y , $f^{-1}(H)$ is D -open set in X .
- (3) For any P -closed set M in Y , $f^{-1}(M)$ is D -closed set in X .

Recall, [5], every D-open set is a D-preopen set. For a D-metric space (X, D) and $G \subseteq X$, $Cl_P^D(G)$ is a D-preclosed set and $Int_P^D(G)$ is D-preopen set in D-metric space (X, D) . For a D-metric space (X, D) and $G \subseteq X$, $Cl_P^D(G)$ and $Int_P^D(G)$ that $G \subseteq Cl_P^D(G)$ and $Int_P^D(G) \subseteq G$.

THEOREM 2.5. [5]. For a D-metric space (X, D) and $G \subseteq X$, $Int_P^D(G) = G$ if and only if G is a D-preopen set.

THEOREM 2.6. [5]. For a D-metric space (X, D) and $G \subseteq X$, $Cl_P^D(G) = G$ if and only if G is a D-preclosed set.

DEFINITION 2.7. [6]. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is called D-precontinuous if $f^{-1}(U)$ is a D-preopen set in (X, D) for every D' -open set U in Y .

THEOREM 2.8. [6]. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is a D-precontinuous function if and only if for each $x \in X$ and each D' -open set U in Y with $f(x) \in U$, there exists a D-preopen set V in (X, D) such that $x \in V$ and $f(V) \subseteq U$.

THEOREM 2.9. [6]. Every D-continuous function is D-precontinuous function.

3. CONTRA AND ALMOST D-FUNCTIONS

DEFINITION 3.1. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is called contra D-precontinuous function if $f^{-1}(V)$ is a D-preclosed set in (X, D) for every D' -open set V in Y .

THEOREM 3.2. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is contra D-precontinuous if and only if $f^{-1}(F)$ is a D-preopen set in (X, D) for every D' -closed set F in Y .

PROOF. Suppose that f is contra D-precontinuous. Let G be any D' -closed set in Y . Then $Y - G$ is a D' -open set in Y . Since f is contra D-precontinuous then $X - f^{-1}(G) = f^{-1}(Y - G)$ is a D-preclosed set in (X, D) . That is, $f^{-1}(G)$ is a D-preopen set in (X, D) . Conversely, Let G be any D' -open set in Y . Then $Y - G$ is a D' -closed set in Y . Then by the hypothesis, $f^{-1}(Y - G) = X - f^{-1}(G)$ is a D-preopen set in (X, D) . That is, $f^{-1}(G)$ is a D-preclosed set in (X, D) . Hence f is contra D-precontinuous. \square

THEOREM 3.3. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is contra D-precontinuous if and only if for each $x \in X$ and each D' -closed set G in Y containing $f(x)$, there is a D-preopen set U in (X, D) containing x such that $f(U) \subseteq G$.

PROOF. Suppose that f is contra D-precontinuous. Let $x \in X$ and G be a D' -closed set in Y containing $f(x)$. Then by the last theorem, $U = f^{-1}(G)$ is a D-preopen set in (X, D) . Since $f(x) \in G$ then $x \in f^{-1}(G) = U$ and $f(U) = f(f^{-1}(G)) \subseteq G$. Conversely, Let G be a D' -closed set in Y . For each $x \in f^{-1}(G)$, $f(x) \in G$. Then by the hypothesis, there is a D-preopen set U_x in (X, D) containing x such that $f(U_x) \subseteq G$. Therefore, we obtain

$$f^{-1}(G) = \cup\{U_x : x \in f^{-1}(G)\}.$$

Then $f^{-1}(G)$ is a D-preopen set in (X, D) . Hence by the last theorem, f is a contra D-precontinuous. \square

THEOREM 3.4. The set of all points x in X at which $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces

(X, D) and (Y, D') is not a contra D-precontinuous is identical with the union of the D-frontier of the inverse images of D' -closed sets of Y containing $f(x)$.

PROOF. Suppose that f is not contra D-precontinuous at $x \in X$. Then by Theorem (4.3), there is a D' -closed set G in Y containing $f(x)$ such that $f(U) \not\subseteq G$ for all D-preopen set U in (X, D) . containing x . That is, for all D-preopen set U in (X, D) . containing x , $f(U) \cap (Y - G) \neq \emptyset$ and this implies

$$U \cap f^{-1}(Y - G) \neq \emptyset.$$

Therefore we have

$$x \in Cl_P^D[f^{-1}(Y - G)] = Cl_P^D[X - f^{-1}(G)].$$

However, since $f(x) \in G$, then

$$x \in f^{-1}(G) \subseteq Cl_P^D[f^{-1}(G)].$$

Then

$$x \in Cl_P^D[X - f^{-1}(G)] \cap Cl_P^D[f^{-1}(G)] = \Gamma_P^D[f^{-1}(G)].$$

Conversely, Suppose that $x \in X$ and $x \in \Gamma_P^D[f^{-1}(G)]$ for some D' -closed sets G in Y containing $f(x)$. If f is a contra D-precontinuous at x then there is D-preopen set U in (X, D) . containing x such that $f(U) \subseteq G$. Therefore we have $x \in U \subseteq f^{-1}(G)$. That is,

$$x \in Int_P^D[f^{-1}(G)] \subseteq \Gamma_P^D[f^{-1}(G)].$$

This is a contradiction. Hence by Theorem (3.3), f is not contra D-precontinuous at x . \square

DEFINITION 3.5. The kernel of a subset A of a D-metric space (X, D) is denoted by $Ker(A)$ and dened by $Ker(A) = \cap\{U : A \subseteq U \text{ and } U \text{ is a D-open set in } X\}$.

LEMMA 3.6. Let A and B be a subset of a D-metric space (X, D) . The following hold:

- (1) $x \in Ker(A)$ if and only if $A \cap U \neq \emptyset$ for any D-closed set U containing x .
- (2) $A \subseteq Ker(A)$ and $A=Ker(A)$ if A is a D-open set in X .
- (3) If $A \subseteq B$ then $Ker(A) \subseteq Ker(B)$.

PROOF.

- (1) Let V be any D-closed subset of X , containing x . Suppose that $A \cap V = \emptyset$. Then $A \subseteq X - V$. Since $X - V$ is D-open set, then $Ker(A) \subseteq X - V$. Hence $x \in X - V$ but this is contradiction. Therefore $A \cap V \neq \emptyset$. Conversely, Suppose that $x \notin Ker(A)$. Then there is at least D-open set containing A and $x \notin V$. Then $X - V$ is D-closed set containing x and $A \cap (X - V) = \emptyset$. This is contradiction with the hypothesis. Hence $x \in Ker(A)$.
- (2) By the definition of $Ker(A)$, we get that $A \subseteq Ker(A)$. If A is D-open set, then A is the smallest D-open set containing A . That is, $Ker(A) = A$.
- (3) Let $x \in Ker(A)$. Then by the part (1), $A \cap V \neq \emptyset$ for any D-closed set V containing x . Since $A \subseteq B$, then $B \cap V \neq \emptyset$. Hence $x \in Ker(B)$. That is, $Ker(A) \subseteq Ker(B)$.

\square

THEOREM 3.7. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is a contra D-precontinuous if and only if $f[Cl_P^D[(A)]] \subseteq Ker[f(A)]$ for every subset A of X .

PROOF. Suppose that f is a contra D-precontinuous. Let A be any subset of X . Let $y \notin Ker[f(A)]$. Then by Lemma (3.6), there is a D' -closed set F in Y containing y such that $f(A) \cap F = \emptyset$. Then $A \cap f^{-1}(F) = \emptyset$. Since F is a D' -closed set in Y and f is a contra D-precontinuous then by Theorem (3.2), $f^{-1}(F)$ is a D-preopen set in (X, D) . Then $X - f^{-1}(F)$ is a D-preclosed set in (X, D) , that is,

$$Cl_P^D[X - f^{-1}(F)] = X - f^{-1}(F).$$

Since $A \cap f^{-1}(F) = \emptyset$, then $A \subseteq X - f^{-1}(F)$, this implies,

$$Cl_P^D(A) \subseteq Cl_P^D[X - f^{-1}(F)] = X - f^{-1}(F).$$

Hence $Cl_P^D(A) \cap f^{-1}(F) = \emptyset$. Then $f[Cl_P^D(A)] \cap F = \emptyset$. Hence $y \notin f[Cl_P^D(A)]$. Therefore $f[Cl_P^D(A)] \subseteq Ker[f(A)]$. Conversely, Suppose that $f[Cl_P^D(A)] \subseteq Ker[f(A)]$ for every subset A of X . Let V be any D' -open subset of Y . Then $f^{-1}(V) \subseteq X$. Then by the hypothesis, $f[Cl_P^D(f^{-1}(V))] \subseteq Ker[f(f^{-1}(V))]$. Since V is a D' -open in Y then by Lemma (3.6),

$$f[Cl_P^D(f^{-1}(V))] \subseteq Ker[f(f^{-1}(V))] \subseteq Ker(V) = V.$$

This implies, $Cl_P^D(f^{-1}(V)) \subseteq f^{-1}(V)$, that is, $f^{-1}(V)$ is a D-preclosed in (X, D) . Hence f is contra D-precontinuous. \square

THEOREM 3.8. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is a contra D-precontinuous if and only if $Cl_P^D[f^{-1}(B)] \subseteq f^{-1}[Ker(B)]$ for every subset B of Y .

PROOF. Suppose that f is a contra D-precontinuous. Let B be any subset of Y . Since $f^{-1}(B)$ is a subset of X and f is a contra D-precontinuous then by the last theorem, $f[Cl_P^D[f^{-1}(B)]] \subseteq Ker[f(f^{-1}(B))]$. This implies, then

$$f[Cl_P^D[f^{-1}(B)]] \subseteq Ker[f(f^{-1}(B))] \subseteq Ker(B).$$

Hence $Cl_P^D[f^{-1}(B)] \subseteq f^{-1}[Ker(B)]$.

Conversely, Suppose that $Cl_P^D[f^{-1}(B)] \subseteq f^{-1}[Ker(B)]$ for every subset B of Y . Let V be any open subset of Y . By the hypothesis and Lemma (3.6),

$$Cl_P^D[f^{-1}(V)] \subseteq f^{-1}[Ker(V)] \subseteq f^{-1}(V).$$

That is, $Cl_P^D[f^{-1}(V)] = f^{-1}(V)$ and so $f^{-1}(V)$ is a D-preclosed in (X, D) . Therefore f is a contra D-precontinuous. \square

DEFINITION 3.9. A subset A of a D-metric space (X, D) is called r-open set if $A = Int_D(Cl_D(A))$. The complement of r-open set called r-closed set. A subset of a D-metric space is called a D-preopen set if it is both D-preopen and D-preclosed set.

DEFINITION 3.10. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is called:

- (1) almost D-precontinuous if for each $x \in X$ and each open set V in Y containing $f(x)$, there is a D-preopen set U in (X, D) containing x such that $f(U) \subseteq Int_{D'}[Cl_{D'}[f(U)]]$.
- (2) almost contra D-precontinuous function if $f^{-1}(V)$ is a D-preclosed set in (X, D) for every r-open set V in Y .
- (3) weakly D-precontinuous function, if for each $x \in X$ and each D' -open set V in Y containing $f(x)$, there is a D-preopen set U in (X, D) containing x such that $f(U) \subseteq Cl_{D'}(V)$.

THEOREM 3.11. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is almost contra D-precontinuous if and only if $f^{-1}(F)$ is a D-preopen set in (X, D) for every r-closed set F in Y .

PROOF. It is clear. \square

THEOREM 3.12. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is almost contra D-precontinuous if and only if for each $x \in X$ and each r-closed set F in Y containing $f(x)$, there is a D-preopen set U in (X, D) containing x such that $f(U) \subseteq F$.

PROOF. Suppose that f is almost contra D-precontinuous. Let $x \in X$ and F be a r-closed set in Y containing $f(x)$. Then by the last theorem, $U = f^{-1}(F)$ is a D-preopen set in (X, D) . Since $f(x) \in F$ then $x \in f^{-1}(F) = U$ and $f(U) = f(f^{-1}(F)) \subseteq F$. Conversely, Let F be a r-closed set in Y . For each $x \in f^{-1}(F)$, $f(x) \in F$. Then by the hypothesis, there is a D-preopen set U_x in (X, D) containing x such that $f(U_x) \subseteq F$. Therefore, we obtain

$$f^{-1}(F) = \{U_x : x \in f^{-1}(F)\}.$$

Then $f^{-1}(F)$ is a D-preopen set in (X, D) . Hence by the last theorem, f is an almost contra D-precontinuous. \square

It is clear that every contra D-precontinuous function is almost contra D-precontinuous, since every r-preopen set is open.

THEOREM 3.13. Every almost contra D-precontinuous function is a weakly D-precontinuous.

PROOF. Let $f : (X, D) \rightarrow (Y, D')$ be almost contra D-precontinuous. Let $x \in X$ be any point in (X, D) and V be any D' -open set in Y containing $f(x)$. Then

$$Cl_{D'}(V) = Cl_{D'}[Int_{D'}(V)] \subseteq Cl_{D'}[Int_{D'}(Cl_{D'}(V))].$$

$$Cl_{D'}[Int_{D'}(Cl_{D'}(V))] \subseteq Cl_{D'}[Cl_{D'}(V)] = Cl_{D'}(V),$$

this implies, $Cl_{D'}(V) = Cl_{D'}[Int_{D'}(Cl_{D'}(V))]$. That is, $Cl_{D'}(V)$ is r-closed set in Y containing $f(x)$. Since f is almost contra D-precontinuous then by Theorem (3.12), there is a D-preopen set U in (X, D) containing x such that $f(U) \subseteq Cl_{D'}(V)$. That is, f is a weakly D-precontinuous. \square

The converse of the last theorem need not be true.

THEOREM 3.14. Let $f : (X, D) \rightarrow (Y, D')$ be a function between two D-metric spaces (X, D) and (Y, D') is almost contra D-precontinuous if and only if for each $x \in X$ and each r-open set V in Y non-containing $f(x)$, there is a D-preclosed set U in (X, D) non-containing x such that $f^{-1}(V) \subseteq U$.

PROOF. Suppose that f is almost contra D-precontinuous. Let $x \in X$ and V be a r-open set in Y non-containing $f(x)$. Then $Y - V$ is a r-closed set in Y containing $f(x)$. Then by Theorem (3.12), there is a D-preopen set G in (X, D) containing x such that $f(G) \subseteq Y - V$. That is, $U = X - G$ is a D-preclosed set in (X, D) non-containing x and so

$$G \subseteq f^{-1}(X - G) \subseteq f^{-1}(Y - V) = X - f^{-1}(V).$$

Hence $f^{-1}(V) \subseteq X - G = U$.

Conversely, Let $x \in X$ any point in X and F be any r-closed set in Y containing $f(x)$. $Y - F$ is r-open set in Y non-containing $f(x)$. Then by the hypothesis, there is a D-preclosed set G in (X, D) non-containing x such that $f^{-1}(Y - F) \subseteq G$. Hence $U = X - G$ is a D-preopen set in (X, D) containing x and

$$f(U) = f(X - G) \subseteq f[X - f^{-1}(Y - F)] = f[f^{-1}(F)] \subseteq F.$$

Then by Theorem (3.12), f is an almost contra D-precontinuous \square

THEOREM 3.15. If a function $f : (X, D) \rightarrow (Y, D')$ is a D-precontinuous and contra D-precontinuous then f is an almost D-precontinuous.

PROOF. Let $x \in X$ be any point in X and V be any D' -open set in Y containing $f(x)$. Since f is contra D-precontinuous and $Cl_{D'}(V)$ be a D' -closed set in Y containing $f(x)$ then by Theorem (3.12), there is a D-preopen set U in (X, D) containing x such that $f(U) \subseteq Cl_{D'}(V)$. Since f is a D-preopen and U is a D-preopen set in (X, D) then $f(U)$ is D' -open set in Y and

$$f(U) = Int_{D'}[f(U)] \subseteq Int_{D'}[Cl_{D'}(f(U))] \subseteq Int_{D'}[Cl_{D'}(V)].$$

This shows that f is an almost D-precontinuous. \square

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